Lecture 8
The First-Scattered Distributed Source Technique

In a previous chapter, we discussed the decomposition of the transport solution into its order-of-scatter components. If the scattering is isotropic or mildly anisotropic, the collided components of the angular flux can be mildly anisotropic even if the uncollided flux is extrememly anisotropic. For such problems it can be useful to treat the uncollided component of the flux analytically, and use an approximate transport method to calculate the collided component of the flux. This is often referred to as the first-scattered distributed source technique because the effective inhomogeneous or distributed source for the collided component of the angular flux is the scattering source due to particles that have scattered for the first time. To demonstrate this, let us consider the following problem

$$\mu \frac{\partial \psi}{\partial x} + \sigma_t \psi = \sigma_s \frac{\phi}{4\pi} \quad , x \in [0, x_0], \tag{1}$$

with boundary conditions,

$$\psi(0, \mu) = \frac{\phi_0}{2\pi} \delta(\mu - 1) \quad , \mu > 0, \tag{2a}$$

$$\psi(x_0, \mu) = 0 \quad , \mu < 0. \tag{2b}$$

In order to decompose this problem into separate uncollided flux and collided flux problems, we first assume that the total flux is the sum of the uncollided and collided fluxes, respectively:

$$\psi = \psi^u + \psi^c \,. \tag{3}$$

Substituting from Eq. (3) into Eq. (1), we get

$$\mu \frac{\partial\left(\psi^u + \psi^c\right)}{\partial x} + \sigma_t\left(\psi^u + \psi^c\right) = \sigma_s \frac{\left(\phi^u + \phi^c\right)}{4\pi} \,. \tag{4}$$

The uncollided flux consists of particles that have never interacted. If we set the scattering source to zero, we will get the uncollided flux solution because any particle that interacts will be effectively absorbed. Thus the uncollided flux satisfies

$$\mu \frac{\partial \psi^u}{\partial x} + \sigma_t \psi^u = 0 \,, \tag{5}$$

$$\psi^u(0,\mu) = \frac{\phi_0}{2\pi}\delta(\mu - 1) \quad , \mu > 0, \tag{6a}$$

$$\psi^u(x_0,\mu) = 0 \quad , \mu < 0. \tag{6b}$$

From Eqs. (4) and (5), it follows that the collided flux satisfies

$$\mu \frac{\partial \psi^c}{\partial x} + \sigma_t \psi^c = \sigma_s \frac{\phi^c}{4\pi} + \sigma_s \frac{\phi^u}{4\pi} \,. \tag{7}$$

$$\psi^c(0,\mu) = 0 \quad , \mu > 0, \tag{8a}$$

$$\psi^c(x_0,\mu) = 0 \quad , \mu < 0. \tag{8b}$$

The uncollided flux is easily obtained analytically:

$$\begin{aligned} \psi^u &= \frac{\phi_0}{2\pi}\delta(\mu - 1)\exp\left(-\sigma_t x\right) \quad , \mu > 0, \\ &= 0 \quad , \mu < 0. \end{aligned} \tag{9}$$

Integrating the uncollided flux over all angles, we get

$$\phi^u = \phi_0 \exp\left(-\sigma_t x\right) \ . \tag{10}$$

Substituting from Eq. (10) into Eq. (7), we obtain

$$\mu \frac{\partial \psi^c}{\partial x} + \sigma_t \psi^c = \sigma_s \frac{\phi^c}{4\pi} + \sigma_s \frac{\phi_0}{4\pi} \exp\left(-\sigma_t x\right) \ . \tag{11}$$

It is clear from Eq. (11) that the contribution to the equation for the collided flux from the uncollided flux takes the form of an effective inhomogeneous or distributed source. It in fact physically represents the scattering source due to particles scattering for the first time. This is the origin of the term, "first-scattered distributed source."

It is also clear from Eqs. (8a), (8b), and (11), that the collided flux is not singular, and is thereby far more ameanable to approximation via diffusion theory than the total flux. Thus we use diffusion theory to calculate the collided flux and further assume diffusive properties for the physical domain. Namely, we assume that $\sigma_t = \sigma_s$ and that $\sigma_t x_0 = 1000$ mean-free-paths. Replacing Eqs. (11), (8a), and (8b), respectively, with their diffusion counterparts, we get

$$-\frac{\partial}{\partial x} D \frac{\partial \phi^c}{\partial x} = \sigma_s \phi_0 \exp\left(-\sigma_t x\right) \quad , x \in [0, x_0], \tag{12}$$

$$\phi^c - 2D \frac{\partial \phi^c}{\partial x} = 0 \quad , \text{at } x = 0, \tag{13a}$$

$$\phi^c + 2D \frac{\partial \phi^c}{\partial x} = 0 \quad , \text{at } x = x_0, \tag{13b}$$

where we have used Marshak boundary conditions and expressed them in extrapolated form. The homogeneous solution to Eq. (12) is linear:

$$\phi_h^c = a + bx \,, \tag{14}$$

and the inhomogeneous solution is

$$\phi_i^c = -\frac{\phi_0 \exp\left(-\sigma_s x\right)}{\sigma_s D} \,. \tag{15}$$

Thus the total solution is expressed in the following form:

$$\phi^c = a + bx - \frac{\phi_0 \exp\left(-\sigma_s x\right)}{\sigma_s D} \,. \tag{16}$$

The two constants, a and b, are determined by the boundary conditions. Substituting from Eq. (16) into Eq. (13a), we get

$$a - 3\phi_0 - 2D\left(b + 3\sigma_s\phi_0\right) = 0 \,,$$

which simplifies to

$$a - 2Db = 5\phi_0 \,. \tag{17}$$

Substituting from Eq. (16) into Eq. (13b), we get

$$a + bx_0 - 3\phi_0 \exp\left(-\sigma_s x_0\right) + 2D\left(b + 3\sigma_s\phi_0 \exp\left(-\sigma_s x_0\right)\right) = 0 \,,$$

which simplifies to

$$a + b\left(x_0 + 2D\right) = \phi_0 \exp\left(-\sigma_s x_0\right) \,. \tag{18}$$

Solving Eqs. (17) and (18), we get

$$a = \phi_0 \left[\frac{5x_0 + 10D + 2D \exp(-\sigma_s x_0)}{x_0 + 4D} \right] , \tag{19a}$$

$$b = -\phi_0 \left[\frac{5 - \exp(-\sigma_s x_0)}{x_0 + 4D} \right] . \tag{19b}$$

Substituting from Eqs. (19a) and (19b) into Eq. (16), we obtain the solution for the collided scalar flux:

$$\phi^c = \phi_0 \left[\frac{5(x_0 - x) + 10D + (x + 2D) \exp(-\sigma_s x_0)}{x_0 + 4D} - 3 \exp(-\sigma_s x_0) \right] . \tag{20}$$

Adding the uncollided and collided scalar flux solutions we obtain the total scalar flux solution:

$$\phi = \phi_0 \left[\frac{5(x_0 - x) + 10D + (x + 2D) \exp(-\sigma_s x_0)}{x_0 + 4D} - 2 \exp(-\sigma_s x_0) \right] . \tag{21}$$

If we use diffusion theory to compute the total flux solution, we get

$$\phi = 4\phi_0 \left[\frac{(x_0 - x) + 2D}{x_0 + 4D} \right] . \tag{22}$$

Because of the large optical thickness of the slab, D and $\exp(-\sigma_s x_0)$ are negligible relative to x_0. Thus the hybrid solution is well represented on the interior of the domain by

$$\phi = 5(1 - x/x_0) , \tag{23}$$

and the pure diffusion solution is similarly well represented by

$$\phi = 4(1 - x/x_0) . \tag{24}$$

A full transport solution for this problem indicates the the interior solution is indeed linear (as it must be since the slab is highly diffusive) with an extrapolated value of five at the left boundary and an extrapolated value of 0 at the right boundary. Thus we find that the first-scattered distributed source strategy yields an essentially exact solution in terms of both the angular and scalar fluxes in the interior of the slab. It does not yield the correct solution for the angular flux in the boundary layer at the left face, but it yields a surprisingly accurate solution for the scalar flux in this region. Since there is no boundary layer at the right face, it also gives an essentially exact solution for the scalar flux solution in this region. However, it cannot give the correct angular flux solution in this region because the the exact vacuum boundary condition cannot be met.