

# Lecture 21

## The Multigroup Energy Discretization

### 1 Introduction

The purpose of this lecture is to describe the multigroup method, which is essentially the only energy discretization technique this is applied to the transport equation. The multigroup method can be presented Petrov-Galerkin method. It is often derived from the transport equation without resorting to basis functions, but this derivation depends upon the assumption of separability of the solution in space and angle. Since such an assumption is rarely met in practice. This is why we have chosen to present the multigroup method as a Petrov-Galerkin method. There are several rather odd aspects of the standard multigroup method. These oddities can be eliminated while leaving the discrete solution invariant, but we have nonetheless chosen to present the standard method.

### 2 Derivation of the Multigroup Method

We begin the derivation of the multigroup method by considering the monoenergetic transport equation with an anisotropic scattering and distributed sources expanded in Legendre

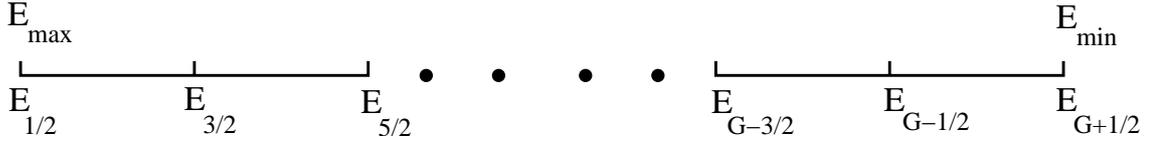


Figure 1: The group structure. Each group represents an interval of energy.

polynomials:

$$\mu \frac{\partial \psi}{\partial x} + \sigma_t \psi = \sum_{k=0}^L \frac{2k+1}{4\pi} \left[ \int_0^\infty \sigma_k(E' \rightarrow E) \phi_k(E') dE' + q_k \right] P_k(\mu), \quad (1)$$

where

$$\sigma_k(E' \rightarrow E) = 2\pi \int_{-1}^{+1} \sigma_s(E' \rightarrow E, \mu_0) P_k(\mu_0) d\mu_0, \quad (2a)$$

$$\phi_k = 2\pi \int_{-1}^{+1} \psi(E) P_k(\mu) d\mu, \quad (2b)$$

$$q_k(E) = 2\pi \int_{-1}^{+1} q(E) P_k(\mu) d\mu. \quad (2c)$$

Next we divide the energy domain,  $[E_{min}, E_{max}]$ , into  $G$  continuous intervals or *groups*, as shown in Fig. 1. One of the oddities of the multigroup method is to index the group energies in descending rather than ascending order. This is done for a reason that is later explained. Thus the edge bounding for group  $g$  are  $E_{g+\frac{1}{2}}$  and  $E_{g-\frac{1}{2}}$ , and the average energy is  $E_g$ . The angular flux trial space representation is given by

$$\tilde{\psi} = \sum_{k=1}^g \psi_k B_k(E), \quad (3)$$

where the basis function,  $B_g$ , is defined for all  $g$  to be zero for all energies not in group  $g$  and of any desired dependence for energies within group  $g$ , subject to the following

normalization:

$$\int_{E_{g+\frac{1}{2}}}^{E_{g-\frac{1}{2}}} B_g(E) dE = 1. \quad (4)$$

Integrating  $\tilde{\psi}$  over group  $g$ , we find that  $\psi_g$  represents the integral of the angular flux over group  $g$ :

$$\psi_g = \int_{E_{g+\frac{1}{2}}}^{E_{g-\frac{1}{2}}} \psi(E) dE. \quad (5)$$

Lacking any knowledge of the behavior of the solution within a group, one generally defines the group basis function to have a constant dependence. The weight functions are defined for all  $g$  as follows:

$$\begin{aligned} W_g(E) &= 0 \quad \text{for all energies not in group } g, \\ &= 1 \quad \text{for all energies in group } g, \end{aligned} \quad (6)$$

The multigroup equations are obtained by substituting the trial space representation for  $\psi$  into Eq. (1), succesively multiplying by each weight function, and integrating the equation over all energies. The resulting equation for  $\psi_g$  is

$$\mu \frac{\partial \psi_g}{\partial x} + \sigma_{t,g} \psi_g = \sum_{k=0}^L \frac{2k+1}{4\pi} \left[ \sum_{n=1}^G \sigma_{k,n \rightarrow g} \phi_{k,n} + q_{k,g} \right] P_k(\mu), \quad (7)$$

where

$$\sigma_{t,g} = \int_{E_{g+\frac{1}{2}}}^{E_{g-\frac{1}{2}}} \sigma_t(E) B_g(E) dE, \quad (8a)$$

$$\sigma_{k,n \rightarrow g} = \int_{E_{g+\frac{1}{2}}}^{E_{g-\frac{1}{2}}} \int_{E_{n+\frac{1}{2}}}^{E_{n-\frac{1}{2}}} \sigma_k(E' \rightarrow E) B_n(E') dE' dE, \quad (8b)$$

$$q_g = \int_{E_{g+\frac{1}{2}}}^{E_{g-\frac{1}{2}}} q(E) B_g(E) dE. \quad (8c)$$

Note from the preceding definitions that  $\sigma_{t,g}$  is obtained from  $\sigma_t$  via a basis-function-weighted average over group  $g$ , that  $\sigma_{k,n \rightarrow g}$  is obtained from  $\sigma_k(E' \rightarrow E)$  via a basis-function-weighted average in  $E'$  over group  $n$  together with integration in  $E$  over group  $g$ , and that  $q_{k,g}$  is obtained from  $q_k$  via a basis-function-weighted average over group  $g$ .

It would perhaps be aesthetically more pleasing if  $\psi_g$  denoted the average of  $\psi$  over group  $g$  rather than the integral of  $\psi$  over group  $g$ . This can be achieved while leaving the solution invariant simply by changing the normalization of the basis functions as follows:

$$\int_{E_{g+\frac{1}{2}}}^{E_{g-\frac{1}{2}}} B_g(E) dE = \Delta E_g, \quad g = 1, G. \quad (9)$$

where  $\Delta E_g = E_{g-\frac{1}{2}} - E_{g+\frac{1}{2}}$ . Furthermore, all multigroup cross-section quantities can be made to represent averages in all variables by changing the definition of the weight functions as follows:

$$W_g(E) = \frac{1}{\Delta E_g} \quad g = 1, G. \quad (10)$$

With these changes, the equation for  $\psi_g$  becomes

$$\mu \frac{\partial \psi_g}{\partial x} + \sigma_{t,g} \psi_g = \sum_{k=0}^L \frac{2k+1}{4\pi} \left[ \sum_{n=1}^G \sigma_{k,n \rightarrow g} \phi_{k,n} \Delta E_n + q_{k,g} \right] P_k(\mu), \quad (11)$$

where

$$\sigma_{t,g} = \frac{1}{\Delta E_g} \int_{E_{g+\frac{1}{2}}}^{E_{g-\frac{1}{2}}} \sigma_t(E) B_g(E) dE, \quad (12a)$$

$$\sigma_{k,n \rightarrow g} = \frac{1}{\Delta E_g} \frac{1}{\Delta E_n} \int_{E_{g+\frac{1}{2}}}^{E_{g-\frac{1}{2}}} \int_{E_{n+\frac{1}{2}}}^{E_{n-\frac{1}{2}}} \sigma_k(E' \rightarrow E) B_n(E') dE' dE, \quad (12b)$$

$$q_g = \frac{1}{\Delta E_g} \int_{E_{g+\frac{1}{2}}}^{E_{g-\frac{1}{2}}} q(E) B_g(E) dE. \quad (12c)$$

We henceforth assume the standard multigroup definitions.

### 3 Solution of the Multigroup $S_n$ Equations

An obvious way to solve the multigroup  $S_n$  equations is simply to use source iteration:

$$\mu \frac{\partial \psi_g^{\ell+1}}{\partial x} + \sigma_{t,g} \psi_g^{\ell+1} = \sum_{k=0}^L \frac{2k+1}{4\pi} \left[ \sum_{n=1}^G \sigma_{k,n \rightarrow g} \phi_{k,n}^\ell + q_{k,g} \right] P_k(\mu). \quad (13)$$

However, this is not necessarily the most effective way to do it. Instead, we use a nested iteration approach. To define this nested scheme, we first decompose the total scattering source as follows:

$$\mu \frac{\partial \psi_g}{\partial x} + \sigma_{t,g} \psi_g = \sum_{k=0}^L \frac{2k+1}{4\pi} \left[ \sum_{n=1}^{g-1} \sigma_{k,n \rightarrow g} \phi_{k,n} + \sigma_{k,g \rightarrow g} \phi_{k,g} + \sum_{n=g+1}^G \sigma_{k,n \rightarrow g} \phi_{k,n} + q_{k,g} \right] P_k(\mu). \quad (14)$$

The first source on the right side of Eq. (14) is the downscatter source for group  $g$ , i.e., the source due to particles in higher-energy groups that scatter into group  $g$  due to interactions that result in energy loss for the scattered particle. The second term in Eq. (14) is the within-group scattering source for group  $g$ , i.e., the source due to particles in group  $g$  that scatter, but do not change group. Such particles may either scatter without energy

loss or suffer such a small energy loss that they stay within the group  $g$ . The third term in Eq. (14) is the upscatter source for group  $g$ , i.e., the source due to particles in lower-energy groups that scatter into group  $g$  due to interactions that result in an energy gain for the scattered particle. Many problems do not involve upscatter. We define our nested iteration procedure so that a minimum number of sweeps are required to solve such problems. In particular, the iteration process can be represented as follows:

$$\mu \frac{\partial \psi_g^{\ell+1, j+1}}{\partial x} + \sigma_{t, g} \psi_g^{\ell+1, j+1} = \sum_{k=0}^L \frac{2k+1}{4\pi} \left[ \sum_{n=1}^{g-1} \sigma_{k, n \rightarrow g} \phi_{k, n}^{\ell, j+1} + \sigma_{k, g \rightarrow g} \phi_{k, g}^{\ell, j} + \sum_{n=g+1}^G \sigma_{k, n \rightarrow g} \phi_{k, n}^{\ell, j} + q_{k, g} \right] P_k(\mu), \quad (15)$$

where  $\ell$  is the inner iteration index and  $j$  is the outer iteration index. This nested iteration process can be described in words as follows:

1. Begin an outer iteration by entering a loop over all groups.
2. For each group,  $g$ , calculate downscatter source using latest fluxes, and calculate the within-group source and the upscatter source using fluxes from the previous outer iteration.
3. Execute the inner iterations by performing sweeps with fixed downscatter and upscatter sources until the within-group scattering source is converged. The outer iteration index for the group  $g$  fluxes is incremented upon convergence of the inner iterations.

4. The outer iteration is completed when the inner iterations are completed for the last group. If there is no upscatter, the converged solution for all groups is obtained after one outer iteration. If there is upscatter, the outer iterations are continued until the upscatter sources are converged for all groups.

To see why only one outer iteration is required when there is no upscatter. Consider that there is no downscatter source for the first group because it is the highest energy group. Thus once the inner iterations are completed for the first group, one has the final solution for that group. The first group then provides the converged downscatter source for the second group. Once the inner iterations are completed for the second group, one has the final solution for that group. The first and second groups then provide the converged downscatter source for the third group, and so on. The reason for indexing the groups in order of decreasing energy is now clear. Assuming that the group fluxes are solved in order of increasing group index, lower group indices must correspond to higher group energies in order to avoid multiple outer iterations when upscatter is not present.