Lecture 14

Equivalence of the S_N and P_{N-1} Methods

The purpose of this lecture is to demonstrate that the 1-D slab-geometry S_N equations with Gauss quadrature and a Legendre cross section expansion of degree $N-1$ are equivalent to the P_{N-1} equations. We begin with the S_N equations:

$$\mu_m \frac{\partial \psi_m}{\partial x} + \sigma_t \psi_m = \sum_{k=0}^{N-1} \frac{2k+1}{4\pi} \left(\sigma_k \phi_k + q_k\right) P_k(\mu_m)\,, \quad m = 1, N. \tag{1}$$

Multiplying Eq. (1) by $P_n(\mu_m)$ and using the following recursion relationship,

$$\mu P_n(\mu) = \frac{n+1}{2n+1} P_{n+1}(\mu) + \frac{n}{2n+1} P_{n-1}(\mu)\,, \tag{2}$$

we obtain

$$\frac{\partial \psi_m}{\partial x} \left[\frac{n+1}{2n+1} P_{n+1}(\mu_m) + \frac{n}{2n+1} P_{n-1}(\mu_m)\right] + \sigma_t \psi_m P_n(\mu_m) =$$
$$\sum_{k=0}^{N-1} \frac{2k+1}{4\pi} \left(\sigma_k \phi_k + q_k\right) P_k(\mu_m) P_n(\mu_m)\,, \quad m = 1, N. \tag{3}$$

Using the Gauss quadrature to integrate Eq. (3) over all directions, we get

$$\frac{n+1}{2n+1} \frac{\partial}{\partial x} \langle \psi P_{n+1} \rangle + \frac{n}{2n+1} \frac{\partial}{\partial x} \langle \psi P_{n-1} \rangle + \sigma_t \langle \psi P_n \rangle =$$
$$\sum_{k=0}^{N-1} \frac{2k+1}{4\pi} \left(\sigma_k \phi_k + q_k\right) \langle P_k P_n \rangle\,, \quad m = 1, N, \tag{4}$$

where $\langle \cdot \rangle$ denotes quadrature integration over angle. Remembering that a Gauss S_N set exactly integrates all polynomials through degree $2N-1$, and that

$$2\pi \int_{-1}^{+1} P_n(\mu) P_k(\mu)\, d\mu = \frac{4\pi}{2k+1} \delta_{n,k}\,, \tag{5}$$

Eq. (4) reduces to the following two equations:

$$\frac{n+1}{2n+1}\frac{\partial \phi_{n+1}}{\partial x} + \frac{n}{2n+1}\frac{\partial \phi_{n-1}}{\partial x} + (\sigma_t - \sigma_n)\, \phi_n = q_n\,, \quad n = 0, N-2, \tag{6a}$$

$$\frac{N}{2N-1}\frac{\partial \phi_N}{\partial x} + \frac{N-1}{2N-1}\frac{\partial \phi_{N-2}}{\partial x} + (\sigma_t - \sigma_{N-1})\, \phi_{N-1} = q_{N-1}\,. \tag{6b}$$

Equations (6a) and (6b) are the P_{N-1} equations provided that $\phi_N = 0$ To demonstrate that this is in fact the case, we note that the Gauss points for an N-point quadrature are the roots of $P_N(\mu)$. Thus,

$$P_N(\mu_m) = 0\,, \quad m = 1, M, \tag{7}$$

and

$$\phi_N = \sum_{m=1}^{N} \psi_m P_N(\mu_m) w_m = 0\,, \tag{8}$$

so Eq. (6b) is actually the correct equation for ϕ_{N-1}:

$$\frac{N-1}{2N-1}\frac{\partial \phi_{N-2}}{\partial x} + (\sigma_t - \sigma_{N-1})\, \phi_{N-1} = q_{N-1}\,. \tag{9}$$

This completes the demonstration that the 1-D slab-geometry S_N equations with Gauss quadrature and a Legendre cross-section expansion of degree $N-1$ are equivalent to the

P_{N-1} equations. However, this does not necessarily imply that the S_N and P_{N-1} equations give the same solutions. This will only be the case if the boundary conditions are equivalent. We have previously discussed only Marshak conditions for the P_n equations. The S_N boundary conditions are equivalent to Mark boundary conditions for the P_{N-1} equations. These conditions are defined for the P_{N-1} equations in terms of the Gauss S_N quadrature points and they require that the exact boundary conditions be approximately met via collocation at the quadrature points. For instance let $\tilde{\psi}$ denote the Legendre expansion for the angular flux associated with the P_{N-1} approximation:

$$\tilde{\psi} = \sum_{n=0}^{N-1} \frac{2n+1}{4\pi} \phi_n P_n(\mu) \,, \tag{10}$$

and let $f(\mu)$ denote the incident flux at a boundary. The Mark reflective condition requires that $\tilde{\psi}(\mu) = \tilde{\psi}(-\mu)$ at the Gauss quadrature points, while the Mark source/vacuum condition requires that $\tilde{\psi}(\mu) = f(\mu)$ at the incoming Gauss quadrature points. As required, the Mark conditions impose $N/2$ equations at each boundary. The Mark and Marshak reflective conditions are equivalent, but the source/vacuum conditions are not equivalent. To give the reader some perspective, we note that the P_1 Marshak source/vacuum condition corresponds to an extrapolation length of $\frac{2}{3\sigma_t}$, while the Mark source/vacuum condition corresponds to an extrapolation length of $\frac{\sqrt{3}}{3\sigma_t}$. The difference between P_n solutions obtained with Marshak conditions and those obtained with Mark conditions rapidly decreases with increasing n.