Lecture 3
Simple Solutions of the 1-D Transport Equation

1 The 1-D Monoenergetic Transport Equation

Consider the 1–D slab geometry, monoenergetic, transport equation with isotropic scattering:

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \sigma_t \psi(x, \mu) = \frac{\sigma_s}{4\pi}\phi(x) + \frac{Q(x)}{4\pi} \ , \tag{1}$$

where

$$\phi = 2\pi \int_{-1}^{+1} \psi(x, \mu) d\mu \tag{2}$$

and $\mu = \cos\theta$, where θ is illustrated in Fig. 1. The boundary conditions for this equation define $\psi(x_L, \mu)$ for $\mu > 0$, and $\psi(x_R, \mu)$ for $\mu < 0$.

1.1 Common Boundary Conditions

Vacuum Boundary Conditions:

$$\psi(x_L, \mu) = 0 \quad , \text{for } \mu > 0.$$

Reflective Boundary Conditions:

$$\psi(x_L, \mu) = \psi(x_L, -\mu) \quad , \text{for } \mu > 0.$$

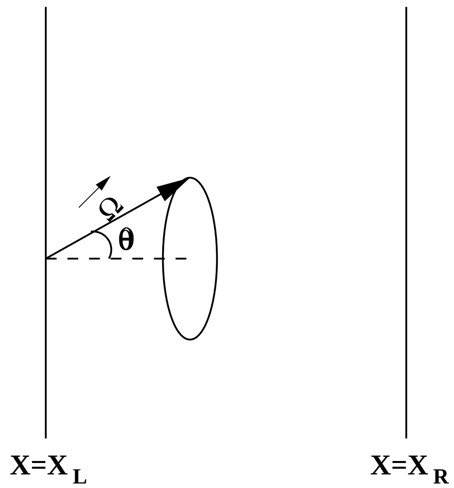

Figure 1: The direction variable, θ. Note that the directional dependence is assumed to be azimuthally symmetric.

Periodic Boundary Conditions:

$$\psi(x_L, \mu) = \psi(x_R, \mu) \quad , \text{for } \mu > 0.$$

Source Boundary Conditions:

$$\psi(x_L, \mu) = \psi(\mu) \quad , \text{for } \mu > 0.$$

2 Pure Absorber Solutions

Consider the following problem.

$$\psi(x_L, \mu) = f(\mu) \quad , \text{for } \mu > 0,$$

$$\psi(x_R, \mu) = 0 \quad , \text{for } \mu < 0,$$

$$\sigma_s = 0\,,$$

$$Q = 0\,.$$

The corresponding transport equation is

$$\mu \frac{\partial \psi}{\partial x} + \sigma_a \psi = 0\,.$$

Dividing the above equation by μ, we get

$$\frac{\partial \psi}{\partial x} + \frac{\sigma_a}{\mu} \psi = 0\,.$$

Note that we have a simple first-order ODE for each value of μ. The solution is a simple exponential. To see this, we first multiply the above equation by $e^{\frac{\sigma_a x}{\mu}}$:

$$e^{\frac{\sigma_a x}{\mu}} \frac{\partial \psi}{\partial x} + e^{\frac{\sigma_a x}{\mu}} \frac{\sigma_a}{\mu} \psi = 0\,.$$

The above equation can be re-expressed as

$$\frac{\partial}{\partial x} \left[e^{\frac{\sigma_a x}{\mu}} \psi(x, \mu) \right]\,.$$

So

$$e^{\frac{\sigma_a x}{\mu}} \psi(x, \mu) = c\,,$$

is a solution, or equivalently,

$$\psi(x, \mu) = c\, e^{-\frac{\sigma_a x}{\mu}}\,.$$

The boundary conditions determine c.

$$\psi(x_L, \mu) = c\ e^{-\frac{\sigma_a}{\mu} x_L} = f(\mu)\,,$$

so

$$c = f(\mu) e^{\frac{\sigma_a}{\mu} x_L}\,.$$

Thus our complete solution is

$$\begin{aligned} \psi(x,\mu) &= f(\mu) e^{-\frac{\sigma_a}{\mu}(x-x_L)} \quad &&, \text{for } \mu > 0, \\ &= 0 &&, \text{for } \mu < 0. \end{aligned} \tag{3}$$

Each ray is exponentially attenuated proportional to the distance that it travels from x_L to x:

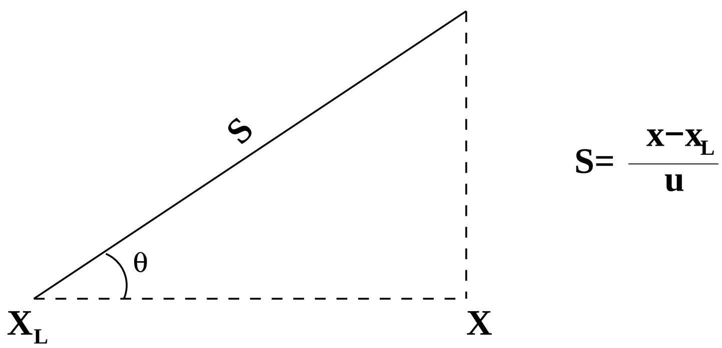

Figure 2: The distance s given x and μ.

3 The Formal Solution with Scattering

Consider the following problem.

$$\psi(x_L,\mu) = f(\mu)\,,$$

$$\psi(x_R,\mu) = g(\mu)\,,$$

$$Q(x) = 0\,.$$

The equation to be solved is

$$\mu\frac{\partial\psi}{\partial x} + \sigma_t\psi = \frac{\sigma_s}{4\pi}\,\phi(x)\,,$$

Use the integrating factor approach again:

$$\frac{\partial\psi}{\partial x} + \frac{\sigma_t}{\mu}\,\psi = \frac{\sigma_s}{4\pi\mu}\,\phi\,,$$

$$e^{\frac{\sigma_t x}{\mu}}\;\frac{\partial\psi}{\partial x} + e^{\frac{\sigma_t x}{\mu}}\;\frac{\sigma_t}{\mu}\;\psi = \frac{\sigma_s}{4\pi\mu}\;e^{\frac{\sigma_t x}{\mu}}\;\phi\,,$$

$$\frac{\partial}{\partial x}\left[e^{\frac{\sigma_t x}{\mu}}\;\psi\right] = \frac{\sigma_s}{4\pi\mu}e^{\frac{\sigma_t x}{\mu}}\;\phi\,,$$

$$\psi(x,\mu)e^{\frac{\sigma_t x}{\mu}} - \psi(x_L,\mu)e^{\frac{\sigma_t x_L}{\mu}} = \int_{x_L}^{x}\frac{\sigma_s}{4\pi\mu}\;e^{\frac{\sigma_t x'}{\mu}}\;\phi(x')dx' \quad \text{, for } \mu > 0,$$

$$\psi(x_R,\mu)e^{\frac{\sigma_t x_R}{\mu}} - \psi(x,\mu)e^{\frac{\sigma_t x}{\mu}} = \int_{x}^{x_R}\frac{\sigma_s}{4\pi\mu}\;e^{\frac{\sigma_t x'}{\mu}}\;\phi(x')dx' \quad \text{, for } \mu < 0,$$

So the final solution is

$$\psi(x,\mu) = \psi(x_L,\mu)e^{-\frac{\sigma_t}{\mu}(x-x_L)} + \int_{x_L}^{x}\frac{\sigma_s}{4\pi\mu}\;e^{\frac{\sigma_t}{\mu}(x'-x)}\,\phi(x')dx' \quad \text{, for } \mu > 0,$$

and

$$\psi(x,\mu) = \psi(x_R,\mu)e^{\frac{\sigma_t}{\mu}(x_R-x)} - \int_x^{x_R} \frac{\sigma_s}{4\pi\mu}\, e^{\frac{\sigma_t}{\mu}(x'-x)}\, \phi(x')dx' \quad ,\text{for } \mu < 0.$$

Note that if you know $\phi(x)$, you need only perform an integral to get the solution, but $\phi(x)$ is actually an integral of $\psi(x,\mu)$. This suggests that you might be able to iterate to a solution. Specifically, you can use the order-of-scatter or Neumann series technique:

First we calculate the uncollided flux:

$$\psi^{(o)}(x,\mu) = \psi(x_L,\mu)e^{-\frac{\sigma_t}{\mu}(x-x_L)} \quad ,\text{for } \mu > 0,$$

and

$$\psi^{(o)}(x,\mu) = \psi(x_R,\mu)e^{\frac{\sigma_t}{\mu}(x_R-x)} \quad ,\text{for } \mu < 0.$$

Next we calculate the first-scattered flux:

$$\psi^{(1)}(x,\mu) = \int_{x_L}^{x} \frac{\sigma_s}{4\pi\mu}\, e^{\frac{\sigma_t}{\mu}(x'-x)}\, \phi^{(o)}(x')dx' \quad ,\text{for } \mu > 0,$$

and

$$\psi^{(1)}(x,\mu) = -\int_x^{x_R} \frac{\sigma_s}{4\pi\mu}\, e^{\frac{\sigma_t}{\mu}(x'-x)}\, \phi^{(o)}(x')dx' \quad ,\text{for } \mu < 0.$$

where

$$\phi^{(o)}(x) = 2\pi \int_{-1}^{+1} \psi^{(o)}(x,\mu)d\mu\,.$$

$\vdots$

Continue on

$\vdots$

You calculate the $n+1$'th-scattered angular flux using the n'th-scattered scalar flux:

$$\psi^{(n+1)} = \int_{x_L}^{x} \frac{\sigma_s}{4\pi\mu} e^{\frac{\sigma_t}{\mu}(x'-x)} \phi^{(n)}(x') dx' \quad , \text{for } \mu > 0,$$

and

$$\psi^{(n+1)} = -\int_{x}^{x_R} \frac{\sigma_s}{4\pi\mu} e^{\frac{\sigma_t}{\mu}(x'-x)} \phi^{(n)}(x') dx' \quad , \ \mu < 0.$$

Finally, add up the contributions to obtain the total angular flux:

$$\psi(x,\mu) = \sum_{n=0}^{\infty} \psi^{(n)}(x,\mu)\,.$$

This is very closely related to the basic iteration technique used in S_n codes.