Appendix C
Eigenfunctions of the Boltzmann Operator

The purpose of this appendix is to show that the spherical-harmonic functions are the eigenfunctions of the the Boltzmann scattering operator, and the composite (removal minus scattering) Boltzmann operator. We begin by defining the spherical-harmonic function of degree l and order m:

$$\begin{aligned} Y_l^m(\overrightarrow{\Omega}) &= \sqrt{C_l^m} P_l^m(\mu)\cos(m\,\Phi) \qquad ,0 \le m \le l, \\ &= \sqrt{C_l^m} P_l^{|m|}(\mu)\sin(|m|\,\Phi) \quad ,-l \le m < 0, \end{aligned} \tag{1}$$

where $P_l^m(x)$ is the associated Legendre function, μ is the cosine of the polar angle, Φ is the azimuthal angle, and

$$C_l^m = (2 - \delta_{m,0})\,\frac{(l-|m|)!}{(l+|m|)!}\,. \tag{2}$$

The spherical-harmonic functions are orthogonal:

$$\int_0^{2\pi}\int Y_l^m\, Y_k^j \; d\mu d\Phi = \delta_{l,k}\delta_{m,j}\frac{4\pi}{2l+1}\,. \tag{3}$$

Furthermore, the space spanned by the harmonics of degree L, where L is any positive integer, is rotationally invariant. This means that any arbitrarily rotated harmonic of degree L can be expressed as a linear combination of harmonics of degree L. This property is unique to the spherical harmonics. There exists no other function spaces defined over the unit sphere that are rotationally invariant.

To demonstrate that the spherical harmonics are eigenfunctions of the composite Boltzmann operator, we apply this operator to an arbitrary spherical-harmonic function, Y_k^j:

$$\begin{aligned}(\sigma_t - S)\, Y_k^j &= \sigma_t Y_k^j - S Y_k^j \\ &= \sigma_t Y_k^j - \int_{4\pi} \sigma_s \left(\overrightarrow{\Omega}' \cdot \overrightarrow{\Omega}\right) \, Y_k^j \left(\overrightarrow{\Omega}'\right) d\Omega' \,. \end{aligned} \tag{4}$$

Next we expand the scattering cross-section in the spherical harmonic functions of order 0, which are more commonly known as the Legendre polynomials:

$$S Y_k^j = \int_{4\pi} \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \sigma_l P_l^0 \left(\overrightarrow{\Omega}' \cdot \overrightarrow{\Omega}\right) \, Y_k^j \left(\overrightarrow{\Omega}'\right) d\Omega' \,, \tag{5}$$

where

$$\sigma_l = 2\pi \int_{-1}^{+1} \sigma_s\left(\mu_0\right) \, P_l^0\left(\mu_0\right) \, d\mu_0 \,. \tag{6}$$

Using the addition theorem to re-express $P_l^0 \left(\overrightarrow{\Omega}' \cdot \overrightarrow{\Omega}\right)$, we obtain:

$$P_l^0 \left(\overrightarrow{\Omega}' \cdot \overrightarrow{\Omega}\right) = \sum_{m=0}^{l} C_l^m P_l^m\left(\mu\right) P_l^m\left(\mu'\right) \cos\left[m\left(\Phi - \Phi'\right)\right] \,. \tag{7}$$

Applying the formula for the cosine of the difference of two angles to Eq. (7), we obtain

$$\begin{aligned} P_l^0 \left(\overrightarrow{\Omega}' \cdot \overrightarrow{\Omega}\right) &= \textstyle\sum_{m=0}^{l} C_l^m P_l^m\left(\mu\right) P_l^m\left(\mu'\right) \left[\cos(m\,\Phi)\cos(m\,\Phi') + \sin(m\,\Phi)\sin(m\,\Phi')\right] \,, \\ &= \textstyle\sum_{m=-l}^{l} Y_l^m \left(\overrightarrow{\Omega}\right) Y_l^m \left(\overrightarrow{\Omega}'\right) \,. \end{aligned} \tag{8}$$

Substituting from Eq. (8) into Eq. (5), we obtain

$$S Y_k^j = \int_{4\pi} \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \sigma_l \sum_{m=-l}^{l} Y_l^m \left(\overrightarrow{\Omega}\right) Y_l^m \left(\overrightarrow{\Omega}'\right) \, Y_k^j \left(\overrightarrow{\Omega}'\right) d\Omega' \,. \tag{9}$$

Using the orthogonality condition expressed by Eq. (3), we find that Eq. (9) reduces to:

$$SY_k^j = \sigma_k Y_k^j \left(\overrightarrow{\Omega} \right) , \tag{10}$$

Equation (10) demonstrates that the spherical-harmonic function Y_k^j is an eigenfunction of the scattering operator with eigenvalue, σ_k. Substituting from Eq. (10) into Eq. (4), we obtain the final result:

$$\left(\sigma_t - S\right) Y_k^j = \left(\sigma_t - \sigma_k\right) Y_k^j \, . \tag{11}$$

Equation (11) shows that the spherical-harmonic function Y_k^j is an eigenfunction of the composite Boltzmann scattering operator with eigenvalue $\sigma_t - \sigma_k$.