Sequential Dynamical Systems With Threshold Functions *

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1 Introduction

A **sequential dynamical system** (SDS) (see [BH+01] and the references therein) consists of an undirected graph $G(V, E)$ where each node $v \in V$ is associated with a Boolean state (s_v) and a *symmetric Boolean function* f_v (called the **local transition function** at v). The inputs to f_v are s_v and the states of all the nodes adjacent to v. In each step of the SDS, the nodes update their state values using their local transition functions in the order specified by a given permutation π of the nodes.[3] A **configuration** of the SDS is an n-tuple $(b_1, b_2, \ldots, b_n)$ where $n = |V|$ and $b_i \in \{0, 1\}$ is the state value of node v_i. The system starts in a specified initial configuration and each step of the SDS produces a (possibly new) configuration.

The original motivation for studying SDSs and their generalization was to develop a computational theory of discrete socio-technical simulations, e.g. transportation and communication systems. However, the above SDS model is closely related to discrete Hopfield networks [FO99], finite Cellular Automata (CA) [Su95] and Communicating Finite State Machines (CFSMs) [BPT91, HKV97]. As a result, many of our lower and upper bounds apply to such models as well (see Section 4).

Summary of results: Analysis problems for SDSs deal with the question of determining whether a given SDS has a specified property. **Reachability** problems are an important class of such analysis problems where the goal is to determine whether a given SDS starting from a specified initial configuration $\mathcal{I}$ ever reaches a specified configuration $\mathcal{C}$. Complexity aspects of reachability problems for SDSs were first studied in [BH+01] where the following dichotomy result was established. Reachability problems for SDSs are, in general, **PSPACE**-complete. However, they can be solved in polynomial time for SDSs where each local transition function is symmetric and monotone, that is, when each node function is a *positive threshold* function. (For any $k \geq 0$, the **positive k-threshold function** has the value 1 iff the number of 1's in the input is at least k.)

This paper extends the results in [BH+01, FO99, BPT91, HKV97, Su95] on SDSs, Hopfield networks and succinctly specified CFSMs in several directions. We discuss the main results here; for applications of these results, see Section 4.

We show that reachability problems for SDSs remain **PSPACE**-complete even when the set of local transition functions contains only positive and negative threshold functions, thus strengthening the dichotomy in [BH+01]. (For any $k \geq 0$, the **negative k-threshold function** has the value 0 iff the number of 1's in the input is at least k.) This result enables us to prove that when *asymmetric* edge weights are permitted, *positive threshold functions alone* suffice to make reachability problems for SDSs **PSPACE**-complete. Moreover, this hardness result holds even when the edge weights are from $\{0, 1\}$. A consequence of this result is that if the local transition functions are monotone (but not necessarily symmetric), the reachability problems for SDSs remain **PSPACE**-complete.

When each local transition function is a positive threshold function, it was shown in [BH+01] that an SDS reaches a fixed point after at most $\lfloor 3m/2 \rfloor$ steps, where m is the number of edges in the underlying graph. Here, using a more refined analysis, we improve the upper bound to $\lceil (m + n + 1)/2 \rceil$.

2 Hardness Results

Our main hardness result is obtained by simulating a special kind of SDS developed in [BH+01]. A precise statement of this result is given below.

Theorem 2.1 *There is a polynomial time reduction from an SDS $\mathcal{S}$ with underlying graph G, set of local transition functions $\mathcal{F}$ and configurations $\mathcal{I}$ and $\mathcal{B}$ for $\mathcal{S}$ to an SDS $\mathcal{S}_1$ with underlying graph G_1, set of local transition functions $\mathcal{F}_1$ and configurations $\mathcal{I}_1$ and $\mathcal{B}_1$ for $\mathcal{S}_1$ such that*

1. *Each local transition function in $\mathcal{F}_1$ is either a positive threshold function or a negative threshold function.*

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[3] It is easy to extend the definition to more general functions, update orders, edge/vertex weights and succinct representations; see [BH+01].

2. *$\mathcal{S}$ starting in configuration $\mathcal{I}$ reaches $\mathcal{B}$ iff $\mathcal{S}_1$ starting in configuration $\mathcal{I}_1$ reaches $\mathcal{B}_1$. Moreover, for each t, $\mathcal{S}$ reaches $\mathcal{B}$ in t steps iff $\mathcal{S}_1$ reaches $\mathcal{B}_1$ in $t+1$ steps.*

3. *$\mathcal{S}$ starting in configuration $\mathcal{I}$ reaches a fixed point iff $\mathcal{S}_1$ starting in $\mathcal{I}_1$ reaches a fixed point. Moreover the maximum node degree of $\mathcal{S}_1$ is bounded by a quadratic function of the maximum node degree of $\mathcal{S}$. Finally, the number of nodes of $\mathcal{S}_1$ is bounded by a linear function of the number of edges of $\mathcal{S}$.*

The construction used to prove the above theorem is somewhat intricate. ¿From the above theorem and the **PSPACE**-completeness of the reachability problems for SDSs, it follows that the reachability problems for SDSs with positive and negative threshold functions are also **PSPACE**-complete. Moreover, this hardness result holds even for SDSs in which the maximum node degree is a constant.

The **PSPACE**-hardness of reachability problems for SDSs with asymmetric edge weights and positive threshold local transition functions is established by a reduction from the hardness result for SDSs with positive and negative threshold functions. This result also implies the **PSPACE**-hardness of reachability problems for SDSs with monotone local transition functions.

3 Improved Upper Bound

When each local transition function is a positive threshold function, it was shown in [BH+01] using a potential function argument that the number of steps needed for an SDS to reach a fixed point is at most $\lfloor 3m/2 \rfloor$, where m is the number of edges in the underlying graph. We improve this bound to $\lceil (m+n+1)/2 \rceil$. This improvement relies on the following two main ideas.

1. Reference [BH+01] defines a potential function on the nodes and edges of the underlying graph and shows that the upper bound on the total potential of the SDS for any configuration is $3m+n$. For the same potential function, we show that for any SDS and any configuration that has a *predecessor*, the upper bound on the total potential is $2m+n$. (This bound is tight; that is, there is an SDS and a configuration with a predecessor such that the total potential is equal to $2m+n$.)

2. Call a transition of an SDS from a configuration $\mathcal{C}$ to a configuration $\mathcal{C}_1$ **unidirectional** if *every* state transition needed to obtain $\mathcal{C}_1$ from $\mathcal{C}$ is either 0 to 1 or 1 to 0. We observe that for any SDS in which each local transition function is a positive threshold function, if some step produces a unidirectional transition, then all subsequent steps (until a fixed point is reached) also produce unidirectional transitions.

Using the above ideas and the result in [BH+01] that each state change reduces the potential of the SDS by at least 2, we show that the number of SDS steps in which the potential can change is at most $\lceil (m+n+1)/2 \rceil$.

4 Applications

The above results have the following additional implications.

A. Floreén and Orponen [FO99] leave open the question of proving lower bounds on reachability problems for *sequential* symmetric and asymmetric discrete Hopfield networks. Our results show that sequential systems are as hard as parallel systems, thus *answering their conjecture in the affirmative*. Moreover, in contrast to earlier work, the results hold for edge weights that are either 0 or 1.

B. Using the concept of "local simulation", our results imply that when SDSs, discrete Hopfield networks or CFSMs are specified succinctly (using a hierarchical specification of the concurrent transition systems discussed in [HKV97]), the corresponding reachability problems become **EXPSPACE**-hard. These constitute the first such results for Hopfield networks and *highly restricted* classes of CFSMs. For instance, it is easy to see how a simple counter-like CFSM can represent a threshold function. Our results can thus be viewed as identifying "hard instances" by a judicious combination of the power of individual FSMs, the structure of the interaction graph between the FSMs and the semantics of the mechanism for passing messages between the FSMs.

C. The convergence bound obtained for SDSs with positive threshold functions implies an analogous bound for the convergence time of symmetric sequential discrete Hopfield networks with no edge weights. For this problem, the results summarized in [FO99] imply a bound of approximately $3m/2$. Our results improve the bound to $\lceil (m+n+1)/2 \rceil$.

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